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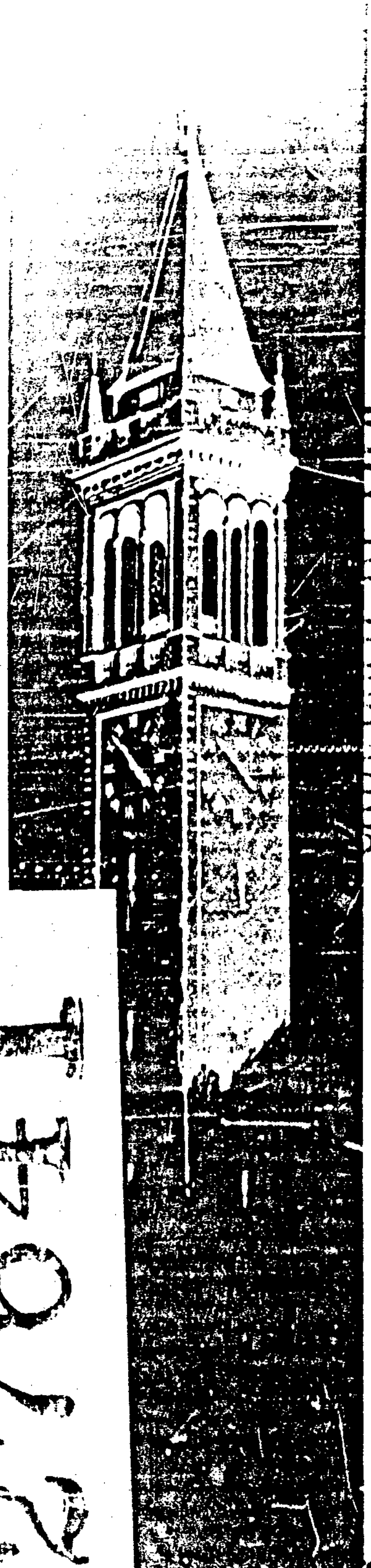

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A Solution to the Frequency-Independent Antenna Problem*

B. R.-S. CHEO†, MEMBER, IRE, V. H. RUMSEY‡, FELLOW, IRE, AND W. J. WELCH‡, MEMBER, IRE

Summary—A solution of Maxwell's equations is obtained for an antenna consisting of an infinite number of equally spaced wires in the form of coplanar equiangular spirals. Radiation amplitude patterns obtained from this solution agree closely with measurements on two-element spiral antennas. The phase pattern shows the approximate validity of a phase center at a distance behind the antenna which decreases with the tightness of the spiral. The current distribution clearly shows increased attenuation with increase in the tightness of the spiral, thus showing how the frequency-independent mode depends on the curvature. A remarkable feature of the solution is that the current consists of an inward traveling wave at infinity when the antenna is excited in that sense which produces an outward wave at the center.

I. Introduction

It has been found in recent years that there is a large class of antennas which are independent of frequency in essentially all their characteristics such as impedance, pattern, polarization and so on.[1-3] The equiangular spiral antenna is one of the basic types: that illustrated in Fig. 1 consists of two conductors cut out of a plane metal sheet. Let us consider how this antenna scales with the wavelength. The shape of the antenna is given by the formula (in polar coordinates r and ϕ)

$$r = e^{-a\phi} \qquad (a \text{ is a constant}). \tag{1}$$

Therefore,

$$\frac{r}{\lambda} = e^{-a(\phi-\phi_0)}, \tag{2}$$

where

$$\phi_0 = \frac{1}{a} \ln \lambda. \tag{3}$$

This shows that a change of wavelength λ is equivalent to turning the antenna through the angle ϕ_0, except for the scaling of the radius r_0 shown in Fig. 1. Now the remarkable property of these antennas is that, so long as the wavelength is shorter than about $2r_0$, the performance is independent of frequency, except for the rotation described in (2) and (3), and therefore it is the same as if r_0 were infinite. Evidently this means that the current distribution must decrease with distance from the input much more rapidly than it does for conventional antennas.

To bring out this point let us compare it with the biconical antenna, shown in Fig. 2. The field, represented by the vectors E and H, decreases as $1/r$ for large values of r, and therefore the surface current J (which equals tangential H) also decreases as $1/r$. The total current I is $2\pi r \sin \alpha J$, where α is the angle of the cone shown in Fig. 2. Thus I remains constant with increasing r. The peculiarity of frequency-independent antennas is then that the field at the surface of the antenna must decrease more rapidly than $1/r$, or alternatively, the total current must decrease fast enough, so that the infinite antenna can be truncated with practically no effect on the radiation pattern.

The theoretical problem posed by the equiangular spiral antenna is to solve Maxwell's equations subject to the vanishing of tangential E on the metal surface, the radiation condition at infinity and the input condition at $r=0$. For the two-element antenna of Fig. 1, this has so far proved intractable even for the infinite case.[4] We are therefore driven to consider some simpler problem which, while retaining the frequency-independent feature, is amenable to theoretical solution. The problem we shall consider in this paper is such a simplification. It can be described by taking an antenna with many elements, as in Fig. 3, the space between the ele-

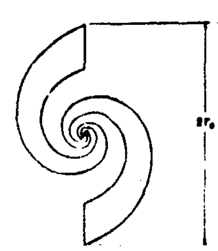

Fig. 1.

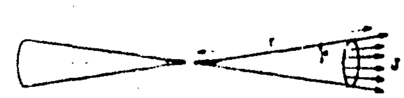

Fig. 2.

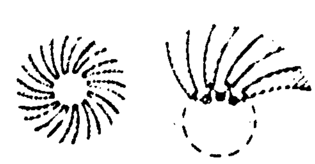

Fig. 3.

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‡ Elec. Engrg. Dept., University of California, Berkeley, Calif.

[1] V. H. Rumsey, "Frequency independent antennas," 1957 IRE National Convention Record, pt. 1, pp. 114-118.

[2] R. H. DuHamel and D. E. Isbell, "Logarithmically periodic antennas," 1957 National Convention Record, pt. 1, pp. 119-128.

[3] J. D. Dyson, "Equiangular spiral antennas," IRE Trans. on Antennas and Propagation, vol. AP-7, pp. 181-187; April, 1959.

[4] P. E. Mast, "A Theoretical Study of the Equiangular Spiral Antenna," Elec. Engrg. Res. Lab., University of Illinois, Urbana, Tech Rept. No. 35; September, 1958.

ments being the same as the space occupied by an element, so that the antenna is "self-complementary" in the sense of Rumsey.[4] We now suppose that the number of elements is infinite, so that the antenna takes the form of a smooth anisotropic sheet which is perfectly conducting in the direction of the spiral lines and perfectly transparent in the perpendicular direction.

This is the kind of problem which can be solved[5] by putting $E=j\eta H$ on one side of the antenna and $E=-j\eta H$ on the other side, where E and H are complex vectors defined according to the $e^{j\omega t}$ time convention, and η is the intrinsic impedance of space. The boundary conditions at the surface are that tangential E be continuous, tangential H be discontinuous by the amount of the surface-current density, E parallel to the spirals be zero, and H parallel to the spirals be continuous. All of these conditions are met if we make E parallel to the wires vanish and tangential E continuous, with $E=j\eta H$ above the surface, and $E=-j\eta H$ below the surface.

The source of fields on this antenna is located at its center. Recognizing that the structure is essentially uniform in azimuth, we assume that the fields of the antenna will have the same dependence on the coordinate ϕ as the source. Thus, we shall take the ϕ variation of the field to be everywhere $e^{jn\phi}$, where n is an integer. This corresponds to the excitation arrangement shown in Fig. 3, in which each generator has the same magnitude as its neighbor and differs infinitesimally from its neighbor in phase. The case $n=1$ corresponds approximately to the excitation of the balanced two-arm antenna, shown in Fig. 1.

II. Formal Solution

Suppose that the antenna lies in the plane $z=0$ of the cylindrical coordinate system of Fig. 4. Let $E_1=j\eta H_1$ for $z>0$ and $E_2=-j\eta H_2$ for $z<0$. Then we have[6]

$$E_1 = -\beta\nabla\times(\hat{z}U_1) + \nabla\times\nabla\times(\hat{z}U_1), \tag{4}$$

$$E_2 = \beta\nabla\times(\hat{z}U_2) + \nabla\times\nabla\times(\hat{z}U_2). \tag{5}$$

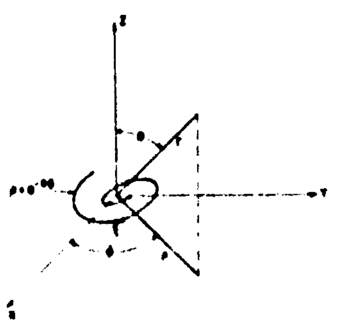

Fig. 4.

[6] V. H. Rumsey, "A New Way of Solving Maxwell's Equations," Electronics Res. Lab., University of California, Berkeley, Series No. 60, Issue No. 345; December 19, 1960. Also to be published in IRE Trans. on Antennas and Propagation.

The functions U_1 and U_2 satisfy the scalar wave equation

$$\nabla^2 U + \beta^2 U = 0, \qquad \beta = \omega/c. \tag{6}$$

We can express a general solution of (6) which varies as $e^{jn\phi}$ by using the Hankel transform formula:

$$U_1 = e^{jn\phi}\int_0^\infty g_1(\lambda)J_n(\lambda\rho)e^{\pm jz\sqrt{\beta^2-\lambda^2}}\lambda d\lambda, \tag{7}$$

$$U_2 = e^{jn\phi}\int_0^\infty g_2(\lambda)J_n(\lambda\rho)e^{\pm jz\sqrt{\beta^2-\lambda^2}}\lambda d\lambda, \tag{8}$$

in which $g_1(\lambda)$ and $g_2(\lambda)$ are arbitrary functions. The Bessel function of the first kind, namely J_n, has been chosen in order that the field be regular at $\rho=0$ for $z\neq 0$. In order that the fields radiate away from the structure, the negative sign must be taken in the exponential factor in the integrand of (7), and the positive sign in the integrand of (8). Then the continuity of tangential electric field at $z=0$ is satisfied if we put $g_1(\lambda)=g_2(\lambda)=g(\lambda)$, as can be verified by direct substitution into (4) and (5). Then,

$$U_1 = e^{jn\phi}\int_0^\infty g(\lambda)J_n(\lambda\rho)e^{-jz\sqrt{\beta^2-\lambda^2}}\lambda d\lambda, \tag{9}$$

$$U_2 = e^{jn\phi}\int_0^\infty g(\lambda)J_n(\lambda\rho)e^{+jz\sqrt{\beta^2-\lambda^2}}\lambda d\lambda. \tag{10}$$

The remaining condition, $E_1\cdot\hat{l}=0$, $\hat{l}$ being tangential to the spiral wires, will determine $g(\lambda)$. From (1) we find $E_1\cdot\hat{l}=0$ implies that

$$aE_{1\rho} = E_{1\phi}. \tag{11}$$

Substitution into this equation from (4) leads to the following expression for the boundary condition:

$$a\left(\frac{\partial^2 U_1}{\partial z\partial\rho} - \frac{\beta}{\rho}\frac{\partial U_1}{\partial\phi}\right) = \frac{1}{\rho}\frac{\partial^2 U_1}{\partial\phi\partial z} + \beta\frac{\partial U_1}{\partial\rho}\bigg|_{z=0}. \tag{12}$$

Then, substituting (9) into (12), we find

$$\int_0^\infty g(\lambda)\left\{(jna\beta + n\sqrt{\beta^2-\lambda^2})\frac{\lambda J_n(\lambda\rho)}{\rho}\right.$$
$$\left. + (ja\sqrt{\beta^2-\lambda^2}+\beta)\lambda^2 J_n'(\lambda\rho)\right\}d\lambda = 0. \tag{13}$$

Then term containing the derivative of the Bessel function may be integrated by parts, so that (13) becomes

$$\int_0^\infty\left\{g(\lambda)(jna\beta + n\sqrt{\beta^2-\lambda^2})\lambda\right.$$
$$\left. - \frac{d}{d\lambda}\left[g(\lambda)\lambda^2(ja\sqrt{\beta^2-\lambda^2}+\beta)\right]\right\}\frac{J_n(\lambda\rho)}{\rho}d\lambda$$
$$+ \frac{g(\lambda)\left[ja\sqrt{\beta^2-\lambda^2}+\beta\right]\lambda^2 J_n(\lambda\rho)}{\rho}\bigg|_0^\infty = 0, \tag{14}$$

Suppose that $g(\lambda)[ja\sqrt{(\beta^2-\lambda^2)}+\beta]\lambda^n J_n(\lambda\rho)$ vanishes for λ equal to zero or infinity. Then the boundary terms in (14) may be discarded. (We shall see later that this assumption is justified.) Applying the inverse Hankel transform to (14) with the boundary terms set equal to zero yields an ordinary differential equation for $g(\lambda)$:

$$(\beta\lambda + j\lambda a\sqrt{\beta^2 - \lambda^2})g'(\lambda)$$

$$+\left[\beta(z-jna)-(n-2ja)\sqrt{\beta^2-\lambda^2}-\frac{ja\lambda^2}{\sqrt{\beta^2-\lambda^2}}\right]g(\lambda)=0. \quad (15)$$

For convenience let $\lambda = y\beta$ and $g(y\beta) = f(y)$. In terms of $f(y)$ the solution to (15) is

$$g(\lambda) = g(y\beta) = f(y) = k\left(\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right)^{n/2}$$

$$\cdot y^{-2}(1+aj\sqrt{1-y^2})^{-1-j(n/a)}. \quad (16)$$

Notice that $f(y)$ is independent of β, exhibiting the frequency-independent nature of the solution explicitly.

For $n>0$, the behavior of $f(y)$ is such that the integral (9) exists, and the assumption that the boundary terms in (14) vanish is valid. For $n<0$, $f(y)$ becomes infinite at $y=0$ or $\lambda=0$ and (9) diverges. It turns out that we can obtain a solution for $n<0$ only if we begin with the assumption that $E_1=-j\eta H_1$ and $E_2=j\eta H_2$. There appears to be a simple explanation for this. With the radiation condition fixed, the choice of the plus or minus sign in the equation $E=\pm j\eta H$ determines the sense of polarization of the far field. At the same time, the sign of n specifies the polarization sense of the source. The interpretation of the situation described above is that the field must have the same sense of polarization as the source.

The complete expressions for U_1 are (taking $n>0$)

$$U_1 = ke^{jn\phi}\int_0^\infty\left(\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right)^{n/2}\frac{(1+aj\sqrt{1-y^2})^{-1-j(n/a)}}{y}$$

$$\cdot e^{-\beta\sqrt{1-y^2}\,\beta z}J_n(\beta\rho y)dy, \quad (17)$$

or, for $n<0$,

$$U_1 = ke^{-jn\phi}\int_0^\infty\left(\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right)^{n/2}\frac{(1-ja\sqrt{1-y^2})^{-1+j(n/a)}}{y}$$

$$\cdot e^{-\beta\sqrt{1-y^2}\,\beta z}J_n(\beta\rho y)dy, \quad (18)$$

where k is a constant which is to be adjusted according to the source strength. Notice that the integrand contains a branch point at $y=+1$ in the complex y plane. The branch cut must be taken in the fourth quadrant, and the path of integration must pass over the branch point in order that $(1-y^2)^{1/2}\to -j(y^2-1)^{1/2}$ for $y>1$. This completes the formal solution to the boundary-value problem.

III. Limiting Cases

In this section we shall evaluate the integral for several limiting cases to find the behavior of the field near the input terminals, the radiation pattern, and the behavior of the antenna current at large distances from the input terminals.

A. The Field Near the Input Terminals

The requirement that the behavior of the field approach the static field distribution near the input terminals was never actually employed in the derivation of the preceding section, and it must be verified that this condition is in fact satisfied by (17) and (18). Let us consider the behavior of the electric field as $\beta r\to 0$. According to (4) and (6),

$$E_1 = -\beta\nabla\times(\hat{z}U_1)+\nabla\times\nabla\times(\hat{z}U_1)$$

$$= -\beta\nabla\times(\hat{z}U_1)+\nabla\left(\frac{\partial U_1}{\partial z}\right)+\beta^2\hat{z}U_1. \quad (19)$$

In the limit as $\beta r\to 0$, the second term of (19) dominates.

$$\lim_{\beta r\to 0} E_1 = \nabla\left(\frac{\partial U_1}{\partial z}\right). \quad (20)$$

This implies that as $\beta r\to 0$, $\partial U/\partial z$ must approach the static potential distribution, which is

$$V = r^{j(n/a)}e^{jn\phi}P_{j(n/a)}^{n}(\cos\theta) = (re^{-a\phi})^{j(n/a)}P_{j(n/a)}^{n}(\cos\theta) \quad (21)$$

The function V satisfies Laplace's equation and is constant along the wires: it is the standard form $r^m P_m^n(\cos\theta)e^{jn\phi}$ with $m=j(n/a)$.

From (17) we find that

$$\frac{\partial U_1}{\partial z} = ke^{jn\phi}\int_0^\infty\left(\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right)^{n/2}\frac{(1+aj\sqrt{1-y^2})^{-1-j(n/a)}}{y}$$

$$\cdot(-j\sqrt{1-y^2})e^{-j\sqrt{1-y^2}\,\beta r\cos\theta}J_n(\beta ry\sin\theta)dy, \quad (22)$$

where we have put $z=r\cos\theta$ and $\rho=r\sin\theta$. For small values of βr, the Bessel function is small except where y is very large, because $J_n(x)\to x^n$ as $x\to 0$. Since the other part of the integrand is well behaved in the neighborhood of $y=0$, the entire integrand contributes very little, except where y is large, in this limit. Therefore, it is reasonable to approximate the part of the integrand other than the Bessel function by its behavior for large y and consider the resulting integral. Hence,

$$\lim_{\beta r\to 0}\frac{\partial U_1}{\partial z} = ke^{jn\phi}\int_0^\infty y^{-1-j(n/a)}e^{-\beta r\cos\theta y}J_n(\beta r\sin\theta y)dy$$

$$= \frac{k\Gamma\left(n-j\frac{n}{a}\right)\Gamma\left(j\frac{n}{a}-n+1\right)}{\Gamma\left(j\frac{n}{a}+n+1\right)}$$

$$\cdot e^{jn\phi}(\beta r)^{j(n/a)}P_{j(n/a)}^{n}(\cos\theta). \quad (23)$$

Apart from the constant multiplier, this agrees precisely with (21).

Furthermore, (23) shows that the magnitude of the current flowing into a sector of the antenna from the source is constant. Thus, if I_i is the current per unit angle at the input, $I_i=\rho J$, where J is the surface current density, and

$$J = 2(H_\rho + aH_\phi)/(1+a^2)^{1/2} = (2/a)(1+a^2)^{1/2}H_\phi.$$

According to (19) and (23),

$$H_\phi\big|_{\pi/2} \propto E_\phi\big|_{\theta=\pi/2} \propto \frac{1}{\rho}e^{jn\phi}(\beta\rho)^{j(n/a)}P_{j(n/a)}{}^{n}(0),$$

and ρ times this quantity has a constant magnitude.

B. Radiation Patterns

In order to investigate the radiation properties of the antenna, we need only consider the asymptotic behavior of the field at large distances from the structure. We shall see that the method of stationary phase readily lends itself to the asymptotic evaluation of (17) for large values of both ρ and z. However, before the integral can be approximated, the differentiations indicated in (5) for the electric field must first be performed. Of interest are the components of the electric field with respect to the spherical coordinate system (r, θ, ϕ) of Fig. 4. Because the distant field is circularly polarized,[5] we need only work out E_ϕ, a component which is common to both the cylindrical and spherical systems. Using (4), we find

$$E_\phi = k\beta^2 e^{jn\phi}\int_0^\infty f(y)\left\{\beta n\sqrt{1-y^2}\,\frac{J_n(\beta y\rho)}{\rho} + \beta^2 y\left[J_{n-1}(\beta\rho y) - \frac{n}{\beta\rho y}J_n(\beta\rho y)\right]\right\}e^{-j\beta z\sqrt{1-y^2}}\,dy, \quad (24)$$

where $\rho=r\sin\theta$ and $z=r\cos\theta$. Except at $\theta=0$, both ρ and z are large when r is large. With ρ large, the leading term in the integrand of (24) is the term containing the factor $J_{n-1}(\beta\rho y)$. Furthermore, the Bessel function may be replaced by its asymptotic value for large argument.[6]

$$\lim_{x\to\infty} J_n(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{e^{j[x-(n+1/2)\pi/2]} + e^{-j[x-(n+1/2)\pi/2]}}{2}\right\}. \quad (25)$$

Using this in the integrand of (24) causes only a second-order error even in the neighborhood of $y=0$, because $f(y)$ tends to zero as y^{n-2} and therefore the integrand tends to zero as y^{2n-1}. Using these approximations and substituting r for ρ sin θ and z cos θ, we obtain the following approximation for (24) for large r:

$$E_\phi \approx k\beta^2 e^{jn\phi}\int_0^\infty \frac{f(y)y^{1/2}}{\sqrt{\pi r\beta\sin\theta}}\cdot\left\{e^{-j\beta r[\cos\theta\sqrt{1-y^2}-y\sin\theta]-j(\pi/2)(n+1/2)} + e^{-j\beta r[\cos\theta\sqrt{1-y^2}+y\sin\theta]+j(\pi/2)(n+1/2)}\right\}dy. \quad (26)$$

This integral contains two terms of the following form: an exponential phase term with a large factor r, multiplied by a relatively slowly-varying function of the variable of integration. According to the principle of stationary phase, the main contribution to this integral comes from the neighborhood of the stationary points of the phase function. In general,[7]

$$\int_a^b g(t)e^{jxh(t)}dt \approx \left[\frac{2\pi}{x\,|h''(\tau)|}\right]^{1/2} g(\tau)e^{jxh(\tau)}e^{\pm j(\pi/4)}, \quad (27)$$

where x is the large parameter, $h'(\tau)=0$, and the plus or minus sign is to be taken according to whether the stationary point is a minimum or maximum. Only the second of the two terms in (26) has a real stationary point, and its value is $y=\sin\theta$. Applying formula (27), we find

$$E_\phi \approx k\beta^2 e^{jn\phi}\cos\theta f(\sin\theta)\frac{e^{-j\beta r}}{r}e^{jn(\pi/2)}. \quad (28)$$

Furthermore, from (16),

$$f(\sin\theta) = \frac{(1+aj\cos\theta)^{-1-j(n/a)}\left(\tan\frac{\theta}{2}\right)^n}{\sin^2\theta}. \quad (29)$$

Finally, the far-zone electric field is

$$E_\phi = k\beta^2\,\frac{\cos\theta(1+aj\cos\theta)^{-1-j(n/a)}\left(\tan\frac{\theta}{2}\right)^n}{\sin^2\theta}\cdot\frac{e^{j[n(\phi+\pi/2)-\beta r]}}{r}, \quad n>0. \quad (30)$$

If we express the field in terms of magnitude and phase

$$E_\phi = A(\theta)e^{-j\psi(\theta)}\frac{e^{j[n(\phi+(\pi/2))-\beta r]}}{r}, \quad (31)$$

we have

$$A(\theta) = \frac{k\beta^2\cos\theta\left(\tan\frac{\theta}{2}\right)^n e^{(n/a)\tan^{-1}(a\cos\theta)}}{\sin\theta\sqrt{1+a^2\cos^2\theta}}, \quad (32)$$

and

$$\psi(\theta) = \frac{n}{2a}\ln|1+a^2\cos^2\theta| + \tan^{-1}a\cos\theta. \quad (33)$$

[6] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 6, p. 359, (18); 1941.

[7] A. Erdelyi, "Asymptotic Expansions," Dover Publications, Inc., New York, N. Y., p. 51, 2.9(2); 1956.

For $e^{-jn\phi}$ excitation $A(\theta)$ is the same but the sign of $\psi(\theta)$ is reversed. The pattern $A(\theta)$ is plotted in Figs. 5 and 6 for various values of n and a. As is typical with frequency-independent antennas, there is no radiation along the surface of the structure. The patterns predicted by (33) agree remarkably well with measurements made by Dyson[3] on two-arm spiral antennas. According to (32), making a small decreases the beamwidth, but only up to a point. For the case $n=1$, the minimum beamwidth attainable is approximately 70°.

Before leaving the discussion of the radiation field, we shall consider the question of whether the antenna has a phase center. The total phase of the far field, apart from the ϕ dependence and some constants, is given by

$$\beta r + \psi(\theta). \tag{34}$$

Because of the complicated form of (33), (34) does not, in general, describe a circular phase front. However, when a is small, $\psi(\theta) \approx a \cos\theta$. In this case, the phase fronts are approximately circular, and, according to the diagram of Fig. 7, the antenna has a phase center located $a/2\pi$ wavelengths behind its center.

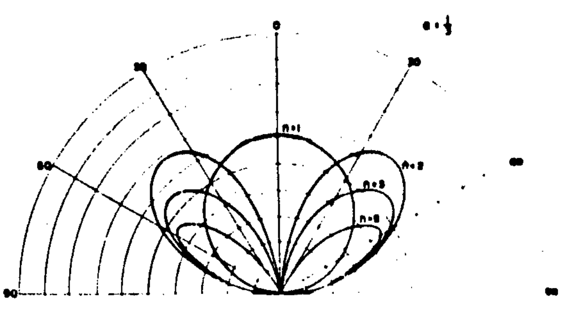

Fig. 5.

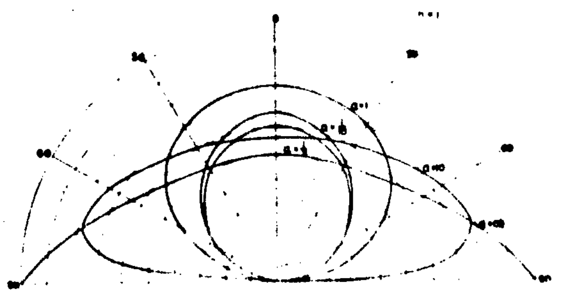

Fig. 6.

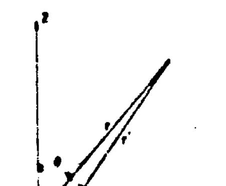

Fig. 7.

C. The Current Distribution

As we saw in Section I, the current distribution is one of the peculiar features of frequency-independent antennas. In the present case, it is obtainable from the field at the plane $z=0$. Since E is proportional to H and E_ϕ is proportional to E_r at $z=0$, the surface-current density is proportional to E_ϕ. The current density per unit of angle ϕ corresponds to the total antenna current I; it varies as ρE_ϕ. Unfortunately it has not been possible to work out the current for all values of ρ. However, fairly simple expressions have been obtained for small values of ρ and alternatively for large values of ρ. For small values of ρ we have already found that

$$I \approx e^{jn(\phi+(1/a)\ln\rho)},$$

which has constant amplitude and rapidly varying phase as a function of ρ. Note however that the phase is constant if we move along a spiral as it ought to be for the steady-current case.

Turning now to the case where ρ is large, according to (4) and (9) and (16), we find that

$$I \propto E_\phi = k\beta^2 e^{jn\phi} \int_0^\infty f(y) \left[n\sqrt{1-y^2}\,\frac{J_n(\beta y\rho)}{\rho} + \beta y J_n'(\beta y\rho) \right] y\,dy. \tag{35}$$

Upon integrating (35) by parts and substituting for $f(y)$ from (16), we obtain, with $n>0$,

$$I \propto e^{jn\phi} \int_0^\infty \left(\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}} \right)^{n/2} \left(n + \frac{1}{\sqrt{1-y^2}} \right) \cdot (1 + aj\sqrt{1-y^2})^{-2-j(n/a)} J_n(\beta\rho y)\, y\,dy. \tag{36}$$

For $n<0$ the correct formula is the conjugate of (36), not the result of reversing the sign of n—see (18). We express the integral as the sum of two integrals over the intervals (0, 1) and (1, ∞), and treat the two parts separately. Consider first the integration over (0, 1):

$$I_1 = \int_0^1 \left(\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}} \right)^{n/2} \left(n + \frac{1}{\sqrt{1-y^2}} \right) \cdot (1 + aj\sqrt{1-y^2})^{-2-j(n/a)} J_n(\beta\rho y)\, y\,dy. \tag{37}$$

The singularity at $y=1$ makes the major contribution to the integral. This is especially true for large $\beta\rho$, in which case the Bessel function oscillates very rapidly as a function of y and cancels all contributions to the integral, except from those regions where the rest of the integrand is also a rapidly varying function of y. We will expand the integrand, excepting the Bessel function, in ascending powers of $(1-y^2)^{1/2}$, beginning with $1/(1-y^2)^{1/2}$, and integrate term by term. Each term in

the resulting series will have successively less importance for large $\beta\rho$ because of the relative smoothness of the successive powers of $[(1-y^2)^{1/2}]^m$. We rewrite (37) slightly and then perform the expansion according to Maclaurin's formula:

$$I_1=\int_0^1\left\{\frac{1}{y^n}\left(\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\right)^{n/2}\left(n+\frac{1}{\sqrt{1-y^2}}\right)\right.$$
$$\left.\cdot(1+aj\sqrt{1-y^2})^{-2-j(n/a)}\right\}y^{n+1}J_n(\beta\rho y)dy$$
$$=\int_0^1\sum_{m=-1}^{\infty}a_m(\sqrt{1-y^2})^m y^{n+1}J_n(\beta\rho y)dy. \qquad (38)$$

Each term in the series may be integrated by means of Sonine's first formula.[8]

$$J_{\mu+(\nu/2)+1}(z)=\frac{z^{(\nu/2)+1}}{2^{(\nu/2)}\Gamma\left(\frac{\nu}{2}+1\right)}\int_0^{\pi/2}J_\mu(z\sin\theta)\sin^{\mu+1}\theta\cos^{\nu+1}\theta d\theta. \qquad (39)$$

Substituting $y=\sin\theta$ in (38) and using (39), we find

$$I_1=\sum_{m=-1}^{\infty}\frac{a_m2^{m/2}\Gamma\left(1+\frac{m}{2}\right)}{(\beta\rho)^{(m/2)+1}}J_{n+(m/2)+1}(\beta\rho). \qquad (40)$$

Consider next the integration over $(1,\infty)$, which we write in the following form:

$$I_2=\int_1^\infty\left\{\left(\frac{1+j\sqrt{y^2-1}}{1-j\sqrt{y^2-1}}\right)^{n/2}(1+\sqrt{y^2-1})^{-2-j(n/a)}\right.$$
$$\left.\cdot\left(n+\frac{j}{\sqrt{y^2-1}}\right)\right\}J_n(\beta\rho y)ydy. \qquad (41)$$

Following the same reasoning as before, we expand the term in the braces of (41) in a series such that each term can be integrated and, furthermore, such that each term has successively less importance for large $\beta\rho$. In this case, the expansion is in powers of $(y^2-1)^{1/2}/y$.

$$I_2=j\int_1^\infty\sum_{m=0}^{\infty}b_m\left(\frac{\sqrt{y^2-1}}{y}\right)^m\frac{J_n(\beta\rho y)dy}{\sqrt{y^2-1}\,y^{n-1}}. \qquad (42)$$

There appears to be no single integral formula which can be applied to every term in (42), so that each term must be treated separately. In what follows, we shall work out only the first four terms of the series obtaining an asymptotic expansion in $\beta\rho$ up to terms which behave as $O(\beta\rho)^{-3}$. The first is

$$I_{20}=jb_0\int_1^\infty\frac{J_n(\beta\rho y)dy}{\sqrt{y^2-1}\,y^{n-1}}$$
$$=jb_0\int_0^\infty\frac{J_n(\beta\rho\sqrt{1+\xi^2})d\xi}{(1+\xi^2)^{n/2}}, \qquad (43)$$

where $y=(1+\xi^2)^{1/2}$. This may be evaluated by means of Sonine's second formula.[9]

$$\int_0^\infty\frac{J_\nu(a\sqrt{t^2+z^2})t^{2\mu+1}dt}{(t^2+z^2)^{\nu/2}}=\frac{2^\mu\Gamma(\mu+1)}{a^{\mu+1}z^{\nu-\mu-1}}J_{\nu-\mu-1}(az). \qquad (44)$$

Let $z=1$, $\beta\rho=a$, and $\mu=-(\frac{1}{2})$. Then, applying (44) to (43) we obtain

$$I_{20}=\frac{jb_0\Gamma(\frac{1}{2})}{\sqrt{2\beta\rho}}J_{n-1/2}(\beta\rho). \qquad (45)$$

The second term is

$$I_{21}=jb_1\int_1^\infty\frac{J_n(\beta\rho y)dy}{y^n}. \qquad (46)$$

An asymptotic expansion of this integral may be obtained by repeated integration by parts. In general,

$$-\int_1^\infty\frac{J_n(\beta\rho y)dy}{y^p}$$
$$=\frac{1}{\beta\rho}J_{n+1}(\beta\rho)+\frac{p+n+1}{(\beta\rho)^2}J_{n+2}(\beta\rho)+O\left(\frac{1}{\beta\rho}\right)^3, \qquad (47)$$

where $p>\frac{1}{2}$. In principle, one could carry out (47) to as many terms as desired. Thus, for the second term,

$$-I_{21}=jb_1\left\{\frac{J_{n+1}(\beta\rho)}{\beta\rho}+(2n+1)\frac{J_{n+2}(\beta\rho)}{(\beta\rho)^2}\right\}$$
$$+O\left(\frac{1}{\beta\rho}\right)^3. \qquad (48)$$

The third term is

$$I_{22}=jb_2\int_1^\infty\frac{\sqrt{y^2-1}}{y}\frac{J_n(\beta\rho y)dy}{y^n}. \qquad (49)$$

After one integration by parts we find

$$I_{22}=-jb_2\left\{\frac{1}{\beta\rho}\int_1^\infty\frac{J_{n+1}(\beta\rho y)dy}{y^n\sqrt{y^2-1}}\right.$$
$$\left.-\frac{2n+2}{\beta\rho}\int_1^\infty\frac{J_{n+1}(\beta\rho y)\sqrt{y^2-1}\,dy}{y^{n+2}}\right\}. \qquad (50)$$

[8] G. N. Watson, "A Treatise on the Theory of Bessel Functions," Cambridge University Press, Cambridge, Eng., 2nd ed., ch. 12, p. 373; 1952.

[9] Ibid., ch. 13, p. 417.

The first term of (50) may be evaluated by means of Sonine's second formula (44). Repeated integration by parts shows that the second term of (50) is $O(\beta\rho)^{-3}$ and may be discarded. Thus,

$$I_{22} = \frac{-jb_2\Gamma(\frac{1}{2})}{\sqrt{2}(\beta\rho)^{3/2}} J_{n+(1/2)}(\beta\rho) + O\left(\frac{1}{\beta\rho}\right)^3. \tag{51}$$

The fourth term of the series is

$$\begin{aligned} I_{23} &= jb_3 \int_1^\infty \left(\frac{y^2-1}{y^2}\right) \frac{J_n(\beta\rho y)dy}{y^n} \\ &= jb_3 \int_1^\infty \frac{J_n(\beta\rho y)dy}{y^n} - jb_3 \int_1^\infty \frac{J_n(\beta\rho y)dy}{y^{n+2}}. \end{aligned} \tag{52}$$

We may apply the result of (47) to the two terms in (52) to obtain

$$I_{23} = jb_3 \frac{2J_{n+2}(\beta\rho)}{(\beta\rho)^2} + O\left(\frac{1}{\beta\rho}\right)^3. \tag{53}$$

It is possible to show that the next term in (42) contributes only $O(\beta\rho)^{-3}$ to the series. Let the input current per unit angle be I_0. Then taking the first four terms of (40), adding them to (45), (48), (51), and (53), and adjusting the constant of proportionality to the input current, we find

$$I = I_0 \frac{a^n}{n} \left| \frac{\Gamma\left(j\frac{n}{a} + n + 1\right)}{\Gamma\left(n - j\frac{n}{a}\right)\Gamma\left(j\frac{n}{a} - n + 1\right)P_{j(n/a)}{}^n(0)} \right| e^{jn\phi} \left\{ \frac{e^{-j\beta\rho + jn(\pi/2)}}{\beta\rho}\left(j + \frac{n^2 + j5an + 6a^2}{2\beta\rho}\right) - \sqrt{\frac{2}{\pi\beta\rho}}\, \frac{\left(\frac{n}{3} + 3n^2 - \frac{26a^2n}{3} - 2ja - 6jan - \frac{2jn^2a}{3} + 8ja^3\right)}{(\beta\rho)^2} \cos\left(\beta\rho - \frac{n\pi}{2} - \frac{\pi}{4}\right) \right\} + O(\beta\rho)^{-3}. \tag{54}$$

For the case $n=1$, this expression reduces to

$$I = I_0 a\sqrt{1+a^2}\, e^{j\phi} \left\{ \frac{-e^{-j\beta\rho}}{\beta\rho}\left(1 + \frac{5a - j - 6ja^2}{2\beta\rho}\right) - \sqrt{\frac{2}{\pi\beta\rho}}\, \frac{(10 - 26a^2 - 26ja + 24ja^3)\sin\left(\beta\rho - \frac{\pi}{4}\right)}{3(\beta\rho)^2} \right\} + O(\beta\rho)^{-3}. \tag{55}$$

Straight wires are represented by $a = \infty$, but our integral representation (17) fails in this case which therefore has to be considered separately. The solution is fairly simple and gives a distribution of $|I|$ which is constant with ρ, and a phase velocity equal to that of light, as illustrated on the graphs.

The phase characteristic is perhaps the most interesting feature of these results. For $n>0$ it consists of an inward slow wave when r is very small, changing to a fast wave as r increases, which becomes infinitely fast at the point where the phase is a maximum in Fig. 9. Passing beyond this point, we find a fast outward wave which slows down to the velocity of light when $r\to\infty$. For $n<0$ we find the same sequence of changes, except that the direction of the phase velocity is reversed everywhere. The extraordinary feature is that we now have an inward wave at infinity. At first sight this might appear to be physically inadmissible because certainly the power must flow outward at infinity. However, in this case we are not dealing with the ordinary radiation field, namely the field which varies as $1/r$, for this is zero on the antenna when $r=\infty$. Indeed, that such a reversal of the phase velocity is necessary with reversal of n can be quickly seen for small r by working from the requirement that the current along any individual wire must be constant in the quasi-static approximation. Also, when $r=\infty$, the curvature of the spiral becomes negligible and the waves become essentially plane. By using the results of Rumsey,[4] it will be found that solutions for straight wires can be constructed in which the phase velocity is inward on the wires but outward some distance away. It is thus possible to see how the inward wave on the antenna is connected to the outward wave in the radiation field, and to the mode of excitation.

In (54) and (55) the Bessel functions have been replaced by their asymptotic expansions.

The current distribution has also been worked out directly from the integral by using a digital computer for the cases $a=0.1$, 0.5 and 1.0 with $n=1$. The results are plotted in Figs. 8–11 (next page). The salient feature of these graphs is the marked increase in attenuation of the current with increase in the curvature of the spiral.

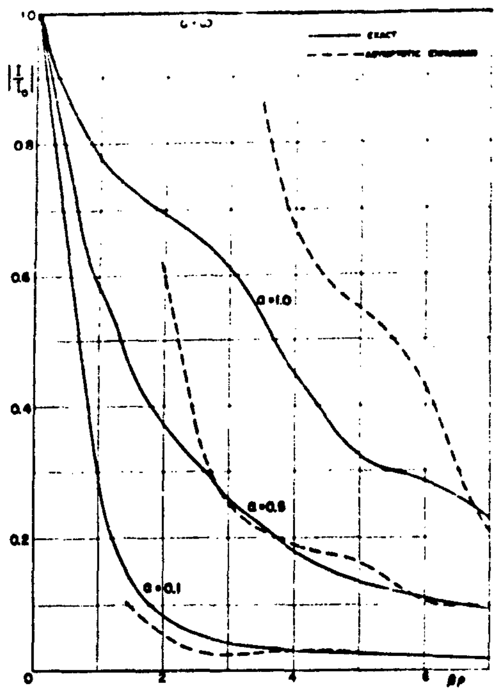

Fig. 8.

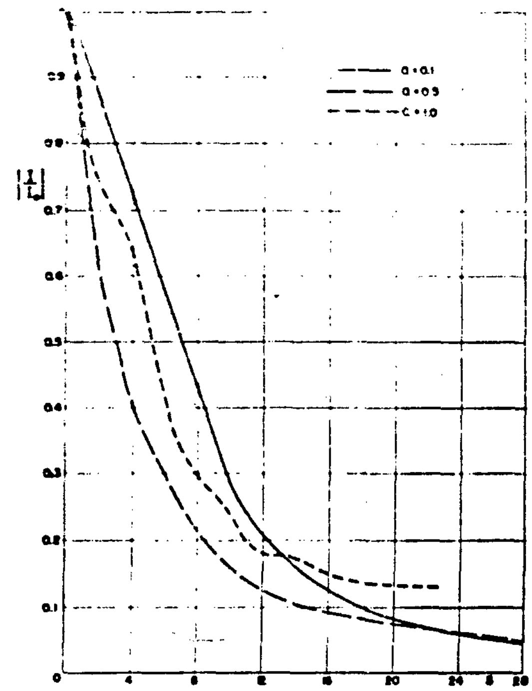

Fig. 10.

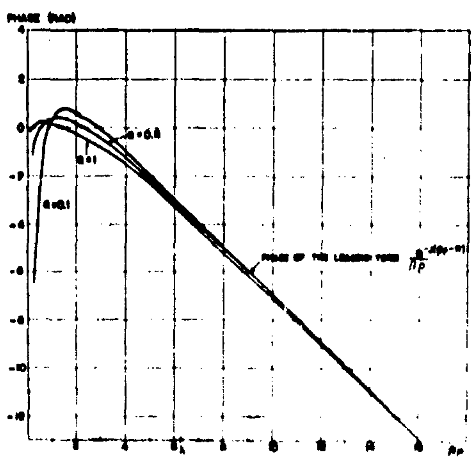

Fig. 9—Phase variation of current distribution as computed by numerical integration.

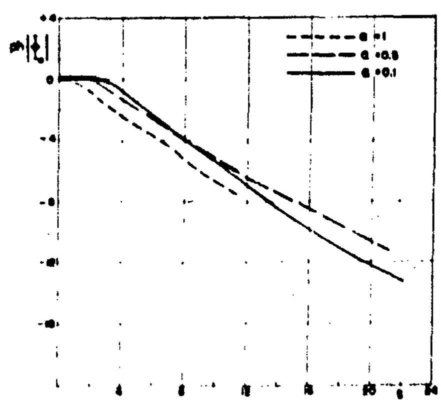

Fig. 11.

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